

Turbulent transport in the solar nebula

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1. Problem Description

It is likely that turbulence played a major role in the evolution of the solar nebula, which is the flattened disk of dust and gas out of which our solar system formed. Relevant turbulent processes include the transport of angular momentum, mass, and heat, which were critically important to the formation of the solar system. This research will break new ground in the modelling of compressible turbulence and its effects on the evolution of the solar nebula. The computational techniques which have been developed should be of interest to researchers studying other astrophysical disk systems (e.g. active galactic nuclei), as well as turbulence modelers outside the astrophysics community.

2. Objectives/Milestones

The work currently being performed system is closely connected with that of Cabot, Hubickyj, and Pollack (hereafter CHP), who have been using the turbulent channel flow code of Kim, Moin, & Moser (1987) to investigate incompressible turbulence in the solar nebula. The turbulence simulation code developed for this project will ultimately replace the central elements of the channel flow code, while incorporating many of the useful analysis tools developed for the latter.

The objectives of this project fall into two categories:

1. To advance the understanding of the role of *compressible* turbulence in the evolution of the solar nebula by carrying out a series of numerical experiments to evaluate the Reynolds stress tensor, turbulent heat transfer rate, turbulent dissipation rate, and turbulent energy spectrum for conditions relevant to the solar nebula;
2. To advance the state of the art in turbulence simulation techniques by exploring new computational methods which are expected to be both efficient and flexible.

The field of solar nebula physics is one of much current interest, as seen, for example, in the contributions to *Protostars and Planets II* (Black & Matthews 1985). The field of turbulence is also one of much current interest.

2.1 Scientific Objectives

The first objective is scientific and largely astrophysical in nature. Turbulent processes are believed to have been important in the evolution of the solar nebula, which was a rarefied disk of gas and dust, out of which the planets, asteroids, and

comets formed. This disk circled the Sun during and shortly after its formation. The nebula is in turn thought to have formed out of the contraction of a much larger and even more diffuse molecular cloud. The nebula's central star, our Sun, also formed out of the molecular cloud matter, and it is the Sun's gravity which held the nebula together and kept it from flying apart. The combination of the Sun's gravitational field, the initial angular momentum of the molecular cloud (retained by the nebular material), and radiative cooling is believed to have confined the solar nebula to a thin disk, rather than a cloud. (The formation and evolution of protostellar disks continue to occur in active star formation regions in our galaxy, and this research is relevant to nebular problems in general, not only to our solar system.)

Anticipated sources of turbulence include thermal convection (driven by temperature gradients), vertical shear in the angular velocity (due to the nonuniform deviations from the central plane orbital motion of the gas with altitude), and the large velocity differential between the rotating disk matter and the infalling (non-Keplerian) molecular cloud material. Turbulent convection is expected to be strongly dependent on the compressibility of the flow; hence, the study of compressible turbulent flow is crucial to this project.

The effects of turbulence on the solar nebula are many. Turbulent dissipation causes a loss of orbital energy and, therefore, an inward flow of mass toward the central star. The inflow of mass must be accompanied by an outflow of specific angular momentum in order to conserve total angular momentum. The inward transport of mass feeds most of the nebular matter to the Sun, while the outward transport of angular momentum causes the nebula to expand. Turbulence will also transport heat, both radially (parallel to the central disk plane) and vertically (in the perpendicular direction). Turbulent dissipation, mentioned above, is expected to be a major (perhaps dominant) source of heat for the nebula. Turbulent dissipation converts kinetic energy to heat, which is then radiated away to space, reducing the total energy of the system and making the nebula more tightly bound to the Sun.

The greatest obstacle to the accurate modeling of the solar nebula is the fact that the length scales on which viscous dissipation takes place (and on which the turbulent kinetic energy is turned into heat) is many orders of magnitude smaller than the size of the disk. Consequently, it is not currently possible to create a single computational model which accurately simulates both the large scale structure and the small scale turbulent dissipative processes. The objective of this project is to simulate numerically the turbulent processes on a small scale and obtain a parameterization of these processes which may be used in other attempts to model the large scale evolution of the nebula. Much of this work is being done by other researchers (CHP), but their efforts must be considered in the developments described below, as the results of their research will be incorporated into this model.

Several issues need to be addressed in order to produce a realistic model for

the solar nebula. The most obvious of these is the Sun's gravity, which forces the gas to follow nearly Keplerian velocity profiles in its rotation about the Sun. Radial shear is, therefore, produced in the flow, as the orbital angular velocity decreases with distance from the Sun.¹

The Keplerian flow velocities are highly supersonic in the rest frame of the central star. A direct simulation of the flow in the star's rest frame is impractical, as the flow velocities dictate unworkably small time steps. A better approach is to work in a coordinate system which is comoving with the average flow in the model volume, so that the velocities are subsonic in the model's coordinate system, and the time steps are more reasonable.

A coordinate transformation based on the work of Rogallo (1981) will be used to represent the radial shear in a form which permits the use of periodic boundary conditions in the radial direction. The Rogallo transformation eliminates the need to devise boundary conditions which properly advect turbulent flow in and out of the radial boundaries.

A second type of shear exists due to the matter falling in from the collapsing molecular cloud onto the disk. The radial and angular velocity components of the infalling material will be quite different from those of the rotating disk. While the vertical velocity component of the infalling matter will be reduced by the accretion shock at the cloud-disk interface, it will remain nonzero, providing a downward directed ram pressure on the disk material. In addition, the radial and angular velocity components of the infalling matter will not be affected by the shock and will create a large velocity shear relative to the disk material. This is an important question, and new techniques may have to be developed to study it. The effects of these factors on nebular turbulence could be important.

A subtler gravitational effect is the vertical variation of gravity within the disk. The disk is very thin compared to its radial dimensions; therefore, a fluid element a distance z above the central disk plane experiences a downward-directed gravitational force (due to the central star) which is proportional to z . This linear variation of gravity is expected to have a significant effect on the convective flow. Convection in the Earth's atmosphere takes place in an altitude range over which the gravitational acceleration is essentially constant. The nebular problem has a variable acceleration with altitude, which may give rise to flow patterns not seen in constant gravity environments and is expected to affect turbulent transport.

Another major issue is the compressibility of the fluid. The computational technique of Kim *et al*, as currently implemented by CHP assumes an incompressible fluid, but the solar nebula is strongly stratified and is, therefore, expected to behave like a compressible gas. Thus it is necessary to use the compressible fluid equations. Most of the work on turbulent flow in the past has

¹ Note: the radial shear by itself is stable and will not directly cause turbulence. Turbulence must be induced by other effects such as viscous heating and vertical shear.

concentrated on incompressible flows.

The channel flow code assumes flow between two parallel plates, for which the fluid velocities go to zero at the plates. The no-slip boundary condition makes sense for flow in a channel but is not valid for flow in the solar nebula. It is desirable to replace these no-slip wall boundary conditions with conditions which are representative of the solar nebula. The nonreflecting boundary conditions of Thompson (1987a) are being used for this purpose. These conditions allow material to flow across the boundaries and allow outward propagating waves to leave the computational domain without generating reflections.

2.2 Improvements to Turbulence Simulation Techniques

The second category of objectives, which may have applications outside of astrophysics (e.g. the aeronautical community), focuses on the improvement of turbulence simulation techniques in general.

The computer code currently in use is very flexible and is able to adjust to a wide variety of boundary conditions and computational techniques. It is designed to be easily modified so that new features may be added and different computational techniques incorporated as needed. For example, the spatial derivative methods can be modified at will (e.g. from finite difference to pseudospectral) by altering only that section of the code which computes these derivatives.

The techniques in use are described in section 3.

2.3 Milestones Passed

The work performed so far has been concerned with analyzing alternative computational methods, developing a suitable computing strategy, and validating the computer simulation. Early design and testing focused on simple linear and nonlinear wave simulations, for which error analysis and convergence testing could be performed easily. The later stages have involved the full set of compressible fluid equations. Successful tests have validated the design of the computer code and are described below. One of these tests has had the added benefit of generating a significant new research project in its own right, as described below.

The major program tests are:

1. Rarefaction wave.
2. Oscillating atmospheric column.
3. Hydrodynamic escape.
4. Unstable shear flow (Kelvin-Helmholtz instability).

2.3.1 Rarefaction Wave

The rarefaction wave is the well known solution to the adiabatic expansion of a uniform gas into a vacuum in the absence of gravity. The exact solution to this problem has been given in many places, (e.g. Landau & Lifshitz 1959,

Thompson 1986). Time dependent numerical simulations of the continuous part of the rarefaction wave were highly accurate and exhibited the proper fourth order convergence with grid refinement.

2.3.2 Oscillating Atmospheric Column

This problem describes a column of atmosphere in a constant gravitational field. The density and pressure drop off exponentially with altitude z , as $e^{-z/h}$, where h is the vertical scale height. This is a one dimensional problem, as the solution does not depend on x or y . The initial state is isothermal and is stable.

Just as in an organ pipe, one can set up standing waves at certain discrete frequencies. I have set up an adiabatic perturbation to excite the lowest frequency mode of the column for problems ranging from 1 to 50 vertical scale heights in extent. In all cases the time dependent solution accurately reproduced the expected oscillatory mode and displayed the correct fourth order convergence. The exponential dependence of the unperturbed state makes this problem an obvious choice for solution by the methods of section 3.1.

2.3.3 Hydrodynamic Escape

This problem is also one dimensional but spherically symmetric. It describes the escape of a planetary atmosphere into space. Gravity obeys the inverse square law, and the flow velocity goes smoothly from highly subsonic near the planet's surface to highly supersonic at large distances. Steady state solutions have been known analytically for some time (Bondi 1952). It is an interesting fact that the same equations describe the solar wind and the accretion of gas onto a protostar embedded in a spherically symmetric cloud.

Although the analytic solution is known, this problem presents special problems for numerical techniques (Zahnle 1988). I have found it to be a challenging problem (and a useful one for uncovering some subtle boundary condition problems) but can now produce stable and accurate solutions. The problem contains enormous pressure and density gradients near the lower boundary, and the density and pressure vary by several orders of magnitude in the model volume. This problem is rendered tractable by the logarithmic techniques described in section 3.1. Previous attempts to solve this problem numerically with more standard methods have failed.

The more general problem of hydrodynamic escape of planetary atmospheres which are affected by thermal conductivity and solar heating is of considerable interest to researchers in the field. The successful solution of the more basic problem will serve as a starting point for further research on this subject by Zahnle, my co-investigators, and myself.

2.3.4 Unstable Shear Flow

This is a well known problem in the atmospheric sciences, and one which is two dimensional in nature. We have an initially isothermal atmosphere whose

density and pressure vary exponentially with altitude z as $e^{-z/h}$ (rectangular coordinates). The atmosphere contains a shear flow with velocity components $(U(z), 0, 0)$, where $U(z) = U_0 \tanh(z/d)$, $d < h$, and U_0 is subsonic. We perturb the flow field by adding small velocity perturbations $u(x, z)$ and $w(x, z)$ so that the perturbed velocity is $(U + u, 0, w)$. Perturbation theory (Chandrasekhar 1961) shows that the stability of the flow is determined by the *Richardson number* $Ri = gd^2/hU_0^2$. A Kelvin-Helmholtz instability arises if $Ri < 1/4$, while the flow is stable if $Ri > 1/4$.

Calculations performed with Richardson numbers of 0.05 and 0.15 do indeed give rise to unstable flow. These calculations show that the initially small perturbations grow and create a vortex. Similar calculations with Richardson numbers of 0.35 and 1.25 are stable and do not lead to vortex formation. These results are consistent with the predictions of linear stability analysis.

3. Technical Approach

The equations to be solved are the compressible Navier Stokes equations. They describe the time evolution of a compressible fluid in three dimensions and incorporate the effects of a variable gravitational field, radial shear, molecular viscosity, thermal conductivity, and internal heat sources. The solution process is made challenging by the large variation in density and pressure which occurs in this problem. A new technique has been developed to handle the range problem as described below.

The purpose of this calculation is to study relatively small scale turbulence in order to characterize the effects of turbulence on the overall flow without resorting to *ad hoc* turbulent viscosity approximations. Therefore, the actual volume to be simulated consists of a small "box" of nebular material in the disk of relatively small radial and angular extent as seen from the Sun.² The box includes one to several scale heights of pressure variation in the direction perpendicular to the disk midplane. Periodic boundary conditions are used at the boundaries facing "sideways" into the disk material (i.e. the ϕ boundaries). The r boundaries make use of Rogallo's method (Rogallo 1981) to specify periodic boundary conditions. The θ (vertical) boundaries require non-trivial boundary conditions and are currently handled by the nonreflecting conditions described by Thompson (1987a).

3.1 The Range Problem

The density and pressure in the nebula fall off very rapidly with altitude above the nebula midplane, roughly as $e^{-(z/h)^2}$, where h is a scale height.³ Thus the density and pressure may vary by orders of magnitude throughout the model volume, which poses a difficult challenge to conventional numerical techniques.

² The spherical coordinate system is assumed for this discussion.

³ The unusually rapid falloff is due to the variable gravity, which increases with altitude.

I have developed a technique to deal with scalar fields which vary over such a large range by using the logarithms of density and pressure instead of the basic quantities themselves. For example, the one dimensional continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0,$$

is replaced by

$$\frac{\partial \ln \rho}{\partial t} + u \frac{\partial \ln \rho}{\partial x} + \frac{\partial u}{\partial x} = 0,$$

and similarly for the energy equation. Thus an exponential dependence of ρ on x is reduced to a polynomial dependence of $\ln \rho$ on x , which the numerical scheme can cope with more readily. All derivatives of density, pressure, and thermal energy may be replaced by their logarithmic forms as above, leading to a more tractable system. An added benefit is that the density and pressure cannot become negative with this approach.

3.1.1 The Numerical Techniques

The current code uses standard fourth order finite difference formulas to evaluate spatial derivatives in all directions (see, for example, Thompson 1987b).⁴

Time integration is handled by the usual fourth order explicit Runge-Kutta method, which is simple to implement and has excellent stability properties. This combination of spatial derivative and time integration methods provides global fourth order convergence. The fourth order convergence rate has been verified repeatedly on a large number of test runs and implies that this code requires significantly fewer grid points than a second order code to achieve a given level of accuracy.

The choice of an explicit method over an implicit method stems from the need to resolve the smallest features present in the flow. At the smallest length scales, viscosity dominates the evolution of the flow. Since we need to simulate the dissipation of kinetic energy to heat accurately at these scales, the grid spacing and time steps necessary are set by the properties of the flow and are the same whether explicit or implicit methods are used. The optimal grid spacing and time step are those for which the propagation and viscous Courant numbers are equal. Consequently, the simpler (and faster) explicit approach has been selected.

⁴ Compact difference formulas of the compact (Padé) type (Rubin & Koshla 1977) have been evaluated but have not been found to offer significant improvements over the standard fourth order formulas.

REFERENCES

- BLACK, D.C. & MATTHEWS, M.S. (EDITORS) 1985 *Protostars and Planets II*, University of Arizona Press, Tucson, Arizona.
- BONDI, H. 1952 On Spherically Symmetrical Accretion. *Monthly Notices of the Royal Astronomical Society*, **112**, No. 2, 195.
- CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press (reprinted by Dover Publications, New York NY).
- KIM, J., MOIN, P. & MOSER, R. 1987 Turbulence statistics in fully developed channel flow at low Reynolds number. *Journal of Fluid Mechanics*, **177**, 133.
- LANDAU, L.D. & LIFSHITZ, E.M. 1959 *Fluid Mechanics*, Pergamon Press, Elmsford NY.
- ROGALLO, R. 1981 *Numerical Experiments in Homogeneous Turbulence*, NASA Technical Memorandum 81315.
- RUBIN, S.G. & KOSHLA, P.K. 1977 Polynomial Interpolation Methods for Viscous Flow Calculations. *Journal of Computational Physics*, **24**, 217.
- THOMPSON, K.W. 1986 The Special Relativistic Shock Tube. *Journal of Fluid Mechanics*, **171**, 365.
- THOMPSON, K.W. 1987a Time Dependent Boundary Conditions for Hyperbolic Systems. *Journal of Computational Physics*, **68**, 1.
- THOMPSON, K.W. 1987b *Lecture Series in Computational Fluid Dynamics*, NASA Technical Memorandum 10010.
- ZAHNLE, K. 1988 Private communication.